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Stochastic gauge transform of the string bundle

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Abstract

We give the notion of frame bundle of the Brownian bridge and we establish over it a Sobolev Calculus. We construct over this frame bundle a non-trivial circle bundle by using its functionals, and we study the associated stochastic gauge transform of this bundle over the original Brownian bridge.

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0. Introduction

Let us consider the based loop space of a two-connected manifold M. It is the space of smooth applications γ_s from the circle S^1 into M such that $\gamma_0 = \gamma_1 = x$. We denote it by $L_x(M)$. Let us introduce a G principal bundle Q over M: we suppose that the structure group is simply connected and compact. We consider the based loop space $L_e(Q)$ of Q starting from e over x: it is a based loop group bundle whose structure group is the based loop $L_e(G)$; its typical element is denoted by q_s and the typical element of the structural based loop group is denoted by g_s. Moreover, $L_e(Q)$ is simply connected.

Let us suppose that G is simple simply laced. There exists a smallest Killing form $\langle\, ,\rangle$ with the following property: the form c over $L_e(G)$ which at the level of the Lie algebra is defined by

$$c(X, Y) = \frac{1}{4\pi^2}\int_0^1 \langle X(s),\, \mathrm{d}Y(s)\rangle \tag{0.1}$$

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is closed Z-valued. Associated to c, there exists a central extension $\tilde{L}_e(G)$ of $L_e(G)$. A string structure is a lift of $L_e(Q)$ by $\tilde{L}_e(G)$, called $\tilde{L}_e(Q)$ [Wi,Ki,Se,CP,CM,ML] Coquereaux and Pilch [CP] and Carey and Murray [CM] have studied geometrically the obstruction to construct a string structure, unlike the algebraic topological considerations of the other references mentioned above. Generally the authors speak of the free loop space, but in this paper, we will work over the based loop space as in [CM], without specifying by notations that it is the based loop space.

Let us introduce the Brownian bridge measure over M, if we suppose that M is a compact Riemannian manifold. The loop $\gamma_.$ in M is only continuous. If we put a connection over Q, the parallel transport τ_s^Q is almost surely defined. A loop $q_.$ over Q can be described as

$$q_s = \tau_s^Q g_s \tag{0.2}$$

with $g_1 = (\tau_1^Q)^{-1}$.

This allows us to construct a measure over $L(Q)$ in two ways:

- In the first way, we construct a measure over the paths of finite energy $g_.$ of G starting from e, and we desintegrate it according $g_1 = (\tau_1^Q)^{-1}$, by using the positivity theorem of [BL] or [AKS]. (The reader can find in [L1] a jump process version). It is the way followed in [L10].
- In the second way, we consider the traditional measure over the path group, which lives over continuous paths, and we desintegrate it according $g_1 = (\tau_1^Q)^{-1}$ (see [Gr3,AM,AH-K,Sh2]).

In the first case, the stochastic fiber of $L(Q)$ is a set of finite energy pinned paths in G. In the second case, the stochastic fiber of $L(Q)$ is a set of continuous pinned paths in G. The first case only is stochastically related to the considerations of [L9,L10].

Carey and Murray [CM] have introduced, when the first Pontryaguin class of Q is equal to 0, a two-cocycle F_Q over $L(Q)$ which allows to construct a circle bundle over $L(Q)$ which gives the string structure $\tilde{L}(Q)$ over $L(Q)$. The transition functions are surely defined.

In the stochastic context, when no measure is defined in the fiber of $L(Q)$, Léandre [L9] has constructed the Hilbert space of sections of spinor fields when a unitary representation of $\tilde{L}(G)$ is given. Unfortunately, $\tilde{L}(Q)$ is only formally a circle bundle over $L(Q)$ when stochastic structures are given in the basis and in the fiber, and so we cannot define $\tilde{L}(Q)$ by the space of sections of $L(Q)$ into $\tilde{L}(Q)$, because generally these sections do not exist, if they are supposed to be regular enough.

In order to overcome this problem, we proceed as before: we define a set of transition functions with values in the circle (instead of $\tilde{L}(G)$ as it was done in [L9,L10]) over $L(Q)$, which are almost surely defined. We define Sobolev spaces over $L(Q)$ such that these functionals with values in the circle belong locally to the first-order Sobolev spaces. This allows us to define formally a circle bundle $\tilde{L}(Q)$ over $L(Q)$, called the stochastic string bundle. We consider the Haar measure over the fiber: it is invariant under rotation. This allows us to define rigorously the space of L^p functionals of $\tilde{L}(Q)$ without to speak of the topological space $\tilde{L}(Q)$ (see [MM] where the topological space of the central extension of a continuous loop group is constructed, but the projection is almost surely defined).

Moreover, we consider a gauge transform of $\tilde{L}(Q)$: it is a measurable functional from $L(M)$ into $\tilde{L}_2(G)$, the space of loops in G with two derivatives bounded in L^2. A gauge transform acts over $L^\infty(\tilde{L}(Q))$. Namely, we have to consider in this case the analogous of the Albeverio–Hoegh-Krohn formulas [AH-K,L10] for C^1 loop group: the Girsanov density is not in all the L^p in this case unlike the case of continuous loop.

If we consider over the fiber the measure which is involved with continuous paths and with the heat kernel over a Lie group [Gr3,AH-K,Sh2,AM], we define $\tilde{L}(Q)$ by its space of functionals. The Albeverio–Hoegh-Krohn density belongs to all the L^p. This allows us to define a gauge transform over $\tilde{L}(Q)$: it is a map from $L(M)$ into $\tilde{L}_1(G)$, the central extension of based loop in G with one derivative in L^2. A gauge transform acts over $L^{\infty-}(\tilde{L}(Q) = \bigcap L^p(\tilde{L}(Q))$.

1. The case of finite energy loop group

We consider a Riemannian compact manifold M. We consider the law of the Brownian bridge $P_{1,x}$ over $L(M)$, the based loop space starting from x. We introduce a G principal bundle Q over M, with compact simply connected group G. We build over Q a connection A^Q. The parallel transport τ_s^Q over a typical loop γ_s in M is almost surely defined with respect to this connection.

We consider $L_{\text{fin}}(Q)$ the space of loop q_s in Q over γ_s of the shape $q_s = \tau_s^Q g_s$ with $g_1 = (\tau_1^Q)^{-1}$ and $g_0 = e$ (see [Bi1,JL1,Dr] for analogous considerations). Following [CM], we can give an abstract meaning to this statement. Let $P_{\text{fin}}(G)$ be the set of finite energy paths in G starting from e. We get a map π from $P_{\text{fin}}(G)$ into G by sending $g_.$ to g_1. $P_{\text{fin}}(G)$ becomes a $L_{\text{fin}}(G)$ principal bundle, this last quantity being the finite energy based loop group of G. We have a map f from $L(M)$ into G: $\gamma_.$ is sent over $(\tau_1^Q)^{-1}$. The map which to $q_.$ associates $(\tau_s^Q)^{-1}q_s$ in $P_{\text{fin}}(G)$ is nothing else than the pullback f^* of f. We get therefore the commutative diagram:

$$\begin{array}{ccc} L_{\text{fin}}(Q) & \to & P_{\text{fin}}(G) \\ \downarrow & & \downarrow \\ L(M) & \to & G \end{array} \tag{1.1}$$

1.1. *First step: Construction of a measure over $L_{\text{fin}}(G)$*

We will begin by constructing following [L10] a measure over $P_{\text{fin}}(G)$. We consider a flat Brownian motion B_s starting from 0 in ϑ, the Lie algebra of G, and C an independent Gaussian variable over ϑ of coariance Id and of average 0. We introduce the following differential equation:

$$\mathrm{d}g_s = g_s(C + B_s)\,\mathrm{d}s, \qquad g_0 = e, \tag{1.2}$$

g_1 has a smooth density $q(g)$ (see [L10]).

Lemma 1.1. $q(g) > 0$.

Proof. We consider the map $(C, B_.) \to g_1$. It belongs to all the Sobolev spaces over the abstract Wiener space defined by the couple $(C, B_.)$, and is surely defined. If we do the small perturbation,

$$(C, B_s) \to (C + \lambda C', B_s + \lambda H_s), \tag{1.3}$$

where H_s has finite energy, we get a perturbated path $g_s(\lambda)$ in G. Moreover, in $\lambda = 0$,

$$\mathrm{d}\frac{\partial}{\partial\lambda} g_s(\lambda) = \frac{\partial}{\partial\lambda} g_s(\lambda)(C + B_s)\,\mathrm{d}s + g_s(C' + H_s)\,\mathrm{d}s. \tag{1.4}$$

Therefore in $\lambda = 0$

$$\frac{\partial}{\partial\lambda} g_1 = \int_0^1 g_u(C' + H_u)g_u^{-1}\,\mathrm{d}u\, g_1. \tag{1.5}$$

Let us study the behavior of the map D

$$(C', H_.) \to \int_0^1 g_u(C' + H_u)g_u^{-1}\,\mathrm{d}u. \tag{1.6}$$

It is equal to

$$\int_0^1 g_u C' g_u^{-1}\,\mathrm{d}u + \int_0^1 \langle K_v, \mathrm{d}/\mathrm{d}v\, H_v\rangle, \tag{1.7}$$

where $K_1 = 0$ and $\mathrm{d}/\mathrm{d}v K_1$ is a rotation of the Lie algebra ϑ of G. We deduce that the image of D for all $(C, B_.)$ is the whole space. By the positivity theorem of [BL,AKS] (see [L1] for a jump version), we deduce that $q(g) > 0$ for all g in G. □

Let us consider $s_1 < s_2 < \cdots < s_n < 1$. $g_{s_1}, \ldots, g_{s_n}, g_1$ has a density $q(g_{s_1}, \ldots, g_{s_n}, g_1)$ (see [L10]). As in [L10], since $q(g) > 0$, we can construct a measure over the C^1 path in G starting from e and arriving in g such that, for any cylindrical functional $F(g_{s_1}, \ldots, g_{s_n})$, we get

$$E_g[F(g_{s_1}, \ldots, g_{s_n})] = \frac{\int F(g_{s_1}, \ldots, g_{s_n})q(g_{s_1}, \ldots, g_{s_n}, g)\,\mathrm{d}\pi g_{s_1}, \ldots, \mathrm{d}\pi g_{s_n}}{q(g)}, \tag{1.8}$$

where $\mathrm{d}\pi g_{s_i}$ is the Haar measure over G. Following the lines of [L10], this gives an expression for the expectation of cylindrical functionals which gives a probability measure over finite energy paths going from e to g. Let us denote this measure by $\mathrm{d}P_g$.

Definition 1.2. Over $L_{\text{fin}}(Q)$, we define the measure μ_{fin} by $\mathrm{d}P_{1,x} \otimes \mathrm{d}P_{(\tau_1^Q)^{-1}}$.

We can give an expression of this measure for the expectation of cylindrical functionals $F(q_{s_1}, \ldots, q_{s_n})$. We get

$$\mu_{\text{fin}}[F(q_{s_1}, \ldots, q_{s_n})] = E_{1,x}[E_{(\tau_1^Q)^{-1}}[F(\tau_{s_1}^Q g_{s_1}, \ldots, \tau_{s_n}^Q g_{s_n})]]. \tag{1.9}$$

1.2. Second step: Construction of Sobolev spaces over $L_{\text{fin}}(Q)$

Let us consider the principal bundle $P_{\text{fin}}(G)$ over G, $g_. \to g_1$. Since G is compact, we can introduce a smooth connection ∇^G for the principal bundle $P_{\text{smooth}}(G) \to G$ where we consider smooth paths g_s. Local sections of this principal bundle are given in the following way: we choose a small neighborhood G_i of G, and we get a smooth section $g_1 \to g_.^i(g_1)$: $(s, g_1) \to g_s^i(g_1)$ is a smooth application from $[0, 1] \times G_i$ into G such that $g_0(g_1) = e$ and such that $g_1^i(g_1) = g_1$. $P_{\text{fin}}(G)$ becomes by that an $L_{\text{smooth}}(G)$ principal bundle.

The connection form over G_i is given by a one-form into G_i with values in $L_{\text{smooth}}(\vartheta)$: let $K_{i,.}(\cdot)(g_1)$ be this connection form.

Let us consider the pullback of this connection form by f, f^* in diagram (1.1). If $(\tau_1^Q)^{-1}$ belongs to G_i, the connection form is given by

$$X \to K_{i,.}(\langle \mathrm{d}(\tau_1^Q)^{-1}, X\rangle)((\tau_1^Q)^{-1}), \tag{1.10}$$

where X is a Bismut's tangent vector over $L(M)$:

$$X_s = \tau_s H_s. \tag{1.11}$$

Namely the tangent space $T_\gamma(L(M))$ is constructed as follows: τ_s is the parallel transport over γ_s for the Levi-Civita connection and H_s is a finite energy loop in $T_{\gamma_0}(M)$ starting from 0, since we consider the based loop space. We put $X_s = \tau_s H_s$ and the set of $X_.$ describes the tangent space of a loop. It is a Hilbert space endowed with the Hilbert structure:

$$\|X\|^2 = \int_0^1 \|\mathrm{d}/\mathrm{d}s\, H_s\|^2 \,\mathrm{d}s < \infty. \tag{1.12}$$

Let us recall that $(\tau_1^Q)^{-1}$ is a group transformation of the fiber of Q in the starting point x. In particular, since $X_0 = X_1 = 0$, we get [Bi1,Gr2]

$$\langle \mathrm{d}(\tau_1^Q), X\rangle = \tau_1^Q \int_0^1 (\tau_s^Q)^{-1} R^Q(\mathrm{d}\gamma_s, X_s)\tau_s^Q, \tag{1.13}$$

where R^Q is the curvature tensor of Q for the connection A^Q.

Let $X_s = \tau_s H_s$, H_s being deterministic, a tangent vector field over the loop space at the basical level M. We deduce a tangent vector X_s^{h} for the based loop space at the total space Q level. If $(\tau_1^Q)^{-1}$ belongs to G_i, it is given by

$$X_s^{\mathrm{h}} = \tau_s H_s - K_{i,s}(\langle \mathrm{d}(\tau_1^Q)^{-1}, X\rangle)((\tau_1^Q)^{-1})g_s, \tag{1.14}$$

where g_s belongs to the loop group fibre: in particular, $g_s^i((\tau_1^Q)^{-1})g_s$ is solution of the stochastic differential equation (1.2) starting from e and arriving at $(\tau_1^Q)^{-1}$. We deduce that g_s is the solution of

$$\mathrm{d}g_s = g_s(C + B_s)\,\mathrm{d}s - (g_s^i)^{-1}\,\mathrm{d}g_s^i\, g_s\,\mathrm{d}s \tag{1.15}$$

with the conditions $g_0 = g_1 = e$. If we do not give the initial condition $g_1 = e$, we can perform the same calculus of variation as in [L10], and we can desintegrate the law of g_s solution of (1.15) according g_1. Its law is absolutely continuous with respect of the law of (1.2) with the same boundary conditions.

Since $K_{i,.}$ is intrinsically defined, X^{h} is intrinsically defined. X^{h} appears as the horizontal lift of the vector X over $L(M)$.

Proposition 1.3. *Let F be a cylindrical functional over $L_{\mathrm{fin}}(Q)$. $F = F(q_{s_1}, \ldots, q_{s_k})$, $s_1 < \cdots < s_k$. There exists a functional* div X^h *which belongs to all the $L^p(\mu_{\mathrm{fin}})$ such that:*

$$\mu_{\mathrm{fin}}[\langle \mathrm{d}F, X^{\mathrm{h}}\rangle] = \mu_{\mathrm{fin}}[F \operatorname{div} X^{\mathrm{h}}]. \tag{1.16}$$

Proof. Modulo a partition of unity over G, we can choose $F(q_{s_1}, \ldots, q_{s_k})h_i((\tau_1^Q)^{-1})$ where h_i is a smooth function from G to [0,1] with support in G_i. $L_{\mathrm{fin}}(Q)$ becomes trivial with this restriction. We split X^{h} into $X + X_1^{\mathrm{v}}$. First we take the derivative of F when τ_1^Q is supposed fixed in the normal direction X_1^{v} and we apply the calculus of [L10]. We get at this stage:

$$E_{(\tau_1^Q)^{-1}}[\langle \mathrm{d}F, X_1^{\mathrm{v}}\rangle] = E_{(\tau_1^Q)^{-1}}[F \operatorname{div} X_1^{\mathrm{v}}], \tag{1.17}$$

where

$$\sup_{(\tau_1^Q)^{-1}\in G_i} E_{(\tau_1^Q)^{-1}}[|\operatorname{div} X_1^{\mathrm{v}}|^p] < \infty. \tag{1.18}$$

Secondly, we take the derivative in the direction X, when we fix an element of the fiber: this means that we take the derivatives of the parallel transport $(\tau_{s_i}^Q)^{-1}$ and not of the g_{s_i}. This belongs to the domain of the traditional integration by parts over the based loop space at the basical level M, because τ_s^Q belongs to all the Sobolev spaces (see [L2]), and in this case there is a divergence associated to $X_s \operatorname{div} X$ which does not depend on the fiber, and which belongs to all the L^p.

We put

$$\operatorname{div} X^{\mathrm{h}} = \operatorname{div} X_1^{\mathrm{v}} + \operatorname{div} X. \qquad \square \tag{1.19}$$

Let $s \to K_s$ be a deterministic loop in ϑ with two derivatives in L^2. Let $X_r^{\mathrm{v}}(K)$ be the vertical vector field associated. We get:

Proposition 1.4. *Let F be a cylindrical functional over $L_{\mathrm{fin}}(Q)$. There exists a functional* $\operatorname{div} X_r^{\mathrm{v}}(K)$ *independent of F which belongs to all the $L^p(\mu_{\mathrm{fin}})$ such that*

$$\mu_{\mathrm{fin}}[\langle \mathrm{d}F, X_r^{v}(K)\rangle] = \mu_{\mathrm{fin}}[F \operatorname{div} X_r^{\mathrm{v}}(K)]. \tag{1.20}$$

Proof. We first operate in the fiber. By using integration by parts formulas over Eq. (1.15) where the vector field is $g_s K_s$ (and not $K_{i,s} g_s$ as it was done in the first case), and after desintegrating them, we deduce that

$$E_{(\tau_1^Q)^{-1}}[\langle \mathrm{d}F, X_r^{\mathrm{v}}(K)\rangle] = E_{(\tau_1^Q)^{-1}}[F \operatorname{div} X_r^{\mathrm{v}}(K)], \tag{1.21}$$

where

$$\sup_{\tau_1^Q} E_{(\tau_1^Q)^{-1}}[|\operatorname{div} X_r^{\mathrm{v}}(K)\ |^p] < \infty \tag{1.22}$$

by repeating the calculus of [L10]. Therefore the result. □

The tangent space of $q_.$ belonging to $L_{\mathrm{fin}}(Q)$ is by definition the sum of vectors $X^{\mathrm{h}} + X_r^{\mathrm{v}}$. We endow it with a Hilbert structure. The vertical part and the horizontal part are orthogonals. For $X^{\mathrm{h}}(H)$, we choose

$$\|X^{\mathrm{h}}(H)\|^2 = \int_0^1 \|\mathrm{d}/\mathrm{d}s\, H_s\|^2\, \mathrm{d}s \tag{1.23}$$

and for $X_r^{\mathrm{v}}(K)$, we choose as in [L10],

$$\|X_r^{\mathrm{v}}(K)\|^2 = \|K_0'\|^2 + \int_0^1 \|K_s''\|^2\, \mathrm{d}s. \tag{1.24}$$

Since we have integration by parts formulas, the notion of derivative can be extended consistently by starting from cylindrical functionals over $L_{\mathrm{fin}}(Q)$. The H-derivative of a functional $\mathrm{d}F$ is a Hilbert–Schmidt cotensor from the Hilbert tangent space $T_{q_.}(L_{\mathrm{fin}}(Q))$ into $\mathbb{R}$. We denote by $\|\mathrm{d}F\|^2$ its Hilbert–Schmidt norm which is almost surely defined.

We put:

Definition 1.5. $W_{1,p}(L_{\mathrm{fin}}(Q))$ is the space of functionals over $L_{\mathrm{fin}}(Q)$ endowed with the norm:

$$\|F\|_{W_{1,p}}\| = \|F\|_{L^p} + \|\ \|\mathrm{d}F\|\ \|_{L^p}. \tag{1.25}$$

1.3. Third step: Functionals over the string bundle

Let us suppose that M is two-connected, such that $L(M)$ is simply connected (see [PS]). Let us suppose that G is simple, simply laced. Therefore $L(Q)$, the space of continuous-based loop of Q, is simply connected.

We will begin by recalling the construction of the string bundle after [CM] when we consider finite energy loop in M. We state the commutative diagram (1.1).

Hypothesis. The first pontryagin class $p_1(Q)$ of Q is equal to 0.

There is a central extension $\tilde{L}_{\mathrm{fin}}(G)$ of $L_{\mathrm{fin}}(G)$ given by a two-cocycle over $L_{\mathrm{fin}}(\vartheta)$, the Lie algebra of $L_{\mathrm{fin}}(G)$:

$$c(X, Y) = \frac{1}{8\pi^2} \int_0^1 \langle X_s, \mathrm{d}Y_s\rangle - \langle Y_s, \mathrm{d}X_s\rangle, \tag{1.26}$$

where X_s and Y_s are finite energy loops in the Lie algebra of G starting from 0. This form fits together into a $P_{\mathrm{fin}}(G)$ two-form: if $X_.$ and $Y_.$ are finite energy paths in ϑ starting from 0,

$$c(X, Y) = \frac{1}{8\pi^2} \int_0^1 \langle X_s, \mathrm{d}Y_s\rangle - \langle Y_s, \mathrm{d}X_s\rangle. \tag{1.27}$$

Since G is simple, all the invariant bilinear forms on ϑ are proportionals. We can find a Killing form such that

$$\omega(X, Y, Z) = \frac{1}{8\pi^2}\langle X, [Y, Z]\rangle \tag{1.28}$$

defines a G invariant three closed-form over G with integral values [PS,CP] because G is simply laced. We choose the smallest one which checks this property.

We get the main result:

$$\mathrm{d}c = \pi^*\omega. \tag{1.29}$$

Since $p_1(Q) = 0$, we get $p_1(Q) = \mathrm{d}\nu$. By [CM],

$$f^*\omega = \mathrm{d} \int_0^1 \nu(\mathrm{d}\gamma_s \cdots) + \mathrm{d}\mu, \tag{1.30}$$

where

$$\mu = \frac{1}{8\pi^2} \int_{0<u<s<1} \langle (\tau_s^Q)^{-1} R^Q(\mathrm{d}\gamma_s, \cdot)\tau_s^Q \wedge (\tau_u^Q)^{-1} R^Q(\mathrm{d}\gamma_u, \cdot)\tau_u^Q\rangle. \tag{1.31}$$

In order to understand this formula, let us recall that μ is a two-form over $L(M)$ which to a couple of vector $(X_., Y_.)$ over $\gamma_.$ associates:

$$\frac{1}{8\pi^2} \int_{0<u<s<1} \langle (\tau_s^Q)^{-1} R^Q(\mathrm{d}\gamma_s, X_s)\tau_s^Q, (\tau_u^Q)^{-1} R^Q(\mathrm{d}\gamma_u, Y_u)\tau_u^Q\rangle + \text{antisymmetry}. \tag{1.32}$$

At this level of computations, we consider finite energy loops in M. But in (1.30)–(1.32), if we consider continuous loop, we can apply the theory of stochastic iterated integrals in order to show that $\int_0^1 \nu(\mathrm{d}\gamma_s \cdots)$ and μ are forms which are almost surely defined (see [JL1]) and that a stochastic exterior derivative is defined over these forms (see [L5,L7,L8]).

Let us come back to the finite energy loop space over M. The upper horizontal map in the commutative diagram (1.1) is the pullback f^* of the horizontal down map f. We consider the form $(f^*)^*c$ over $L(Q)$. It is not closed. Then we perturb it by a basical form and we get $(f^*)^*c - \pi^*(\mu + \nu)$. It becomes a closed form over $L(Q)$, but it is not Z-valued. Let us recall that a Chen form is a finite sum of forms of the type $\int_{0<s_1<\cdots<s_n<1} \omega_1(\mathrm{d}\gamma_{s_1}\cdots)\wedge\cdots\wedge\omega_n(\mathrm{d}\gamma_{s_n}\cdots)$ where the ω_i are forms of degree strictly positive over M. The cohomology of Chen forms is equal to the cohomology of the based loop space [Ad]. We can perturb therefore $(f^*)^*c - \pi^*(\mu + \nu)$ by a closed Chen form $\pi^*\beta$ such that the following property is satisfied: the two-form over $L(Q)$

$$F_Q = (f^*)^*c - \pi^*(\mu + \nu + \beta) \tag{1.33}$$

is closed Z-valued; the integral of F_Q over a surface without boundary in $L(Q)$ is an integer. To F_Q is associated a unique circle bundle over the finite energy based loop space of Q whose curvature is $2\pi\mathrm{i}F_Q$.

F_Q has still a meaning over $L_{\mathrm{fin}}(Q)$. There are two contributions:

- A basical part $-\pi^*(\mu + \nu + \beta)$ which is treated by the theory of stochastic iterated integrals of [JL1].
- A vertical part $(f^*)^*c$, which is a deterministic one. It is of the same shape studied by Carey and Murray [CM], but since the measure over the fiber lives over finite energy paths going from e to $(\tau_1^Q)^{-1}$, the stochastic treatment of this contribution will be analogous to the deterministic treatment of [CM], by adding some more integrability conditions .

In the stochastic context, we will define the circle bundle $\tilde{L}_{\mathrm{fin}}(Q)$ over $L_{\mathrm{fin}}(Q)$ associated to F_Q by its functionals: namely, let us suppose that there exists a set V_i, $i \in N$, of subsets of $L_{\mathrm{fin}}(Q)$ satisfying the following conditions:

(i) $\bigcup V_i = L_{\mathrm{fin}}(Q)$ μ_{fin} almost surely.

(ii) There exists an increasing sequence of functionals G_i^n which belong to all the first-order Sobolev spaces such that almost surely

$$G_i^n > 0 \subset V_i \tag{1.34}$$

and such that almost surely when $n \to \infty$

$$\lim G_i^n = 1_{V_i} \tag{1.35}$$

(iii) Over $V_i \cap V_j$, there exists a random variable $\rho_{i,j}(q.)$ with values in S^1 such that $G_i^n G_j^m \rho_{i,j}(q.)$ belongs to all Sobolev spaces of first order.

(iv) Over $V_i \cap V_j$, μ_{fin} almost surely

$$\rho_{i,j}(q.)\rho_{j,i}(q.) = 1. \tag{1.36}$$

(v) Over $V_i \cap V_j \cap V_k$, μ_{fin} almost surely

$$\rho_{i,j}(q.)\rho_{j,k}(q.)\rho_{k,i}(q.) = 1. \tag{1.37}$$

Remark. Instead of using the notion of a property which is true μ_{fin} almost surely, it should be possible to use the notion of a property which is true quasi-surely. For that, let us define what is a set of capacity 0. We define the capacity of an open set O as the infimum of the first Sobolev norm in L^p of functionals F such that $F \geq 1_O$ almost surely. The infimum of the capacities in L^p of the open set containing a set G is called the capacity in L^p of G. A set is of capacity 0 if all its capacities in L^p are equal to 0.

Definition 1.6. A measurable functional $\tilde{F}(\tilde{q}_.)$ associated to the formal circle bundle constructed from the system of $\rho_{i,j}$ is a family of measurable functionals $F_i : V_i \times S^1 \to \mathbb{R}$ such that almost surely in $q_.$ and u (we take the Haar measure over S^1)

$$F_i(q_., u_i) = F_j(q_., u_j), \tag{1.38}$$

where $u_j = u_i \rho_{i,j}(q_.)$ almost surely.

Let us recall that the Haar measure over the circle is invariant under rotation. If we consider a measurable $\tilde{F}(\tilde{q}_.)$ over the formal circle bundle associated to the system of $\rho_{i,j}$, we can integrate consistently in the circle fiber and we get

$$F_p(q_.) = \int_{\text{fiber}} | \tilde{F}(\tilde{q}_.) |^p \, \mathrm{d}u. \tag{1.39}$$

Definition 1.7. A measurable functional $\tilde{F}(\tilde{q}_.)$ belongs to $L^p(\tilde{\mu}_{\text{fin}})$ if

$$\mu_{\text{fin}}[F_p(q_.)] < \infty. \tag{1.40}$$

Remark. We can go further if there is a connection as in [L10] over the formal circle bundle in order to speak of functionals $\tilde{F}(\tilde{q}_.)$ which belong to some Sobolev spaces associated to the system of $\rho_{i,j}$.

Let us consider the curvature form F_Q over $L_{\text{fin}}(Q)$: we will produce a system of transition maps $\rho_{i,j}(q_.)$ which will satisfy (i)–(v). This will allow us to construct the space of L^p functionals over the formal circle bundle $\tilde{L}_{\text{fin}}(Q)$ over $L_{\text{fin}}(Q)$ called the string bundle of Q.

Let us consider a loop of finite energy in M called γ_i and the ball $B(\gamma_i, \delta)$ for the uniform distance for δ small. We can take a countable set of γ_i such that $\bigcup B(\gamma_i, \delta) = L(M)$. Let $l_{i,t}(\gamma)$ be the path

$$l_{i,t}(\gamma)_s = \exp_{\gamma_{i,s}}[(1-t)(\gamma_s - \gamma_{i,s})] \tag{1.41}$$

joining γ to γ_i. We complete it in any path going from γ_i to γ_{ref}. We produced by that a distinguished path going from $\gamma \in B(\gamma_i, \delta)$ to γ_{ref}: we call it $l_i(\gamma)$. δ is chosen small enough. If γ belongs to $B(\gamma_i, \delta) \cap B(\gamma_j, \delta)$, we can fullfill the triangle $\gamma, \gamma_i, \gamma_j$ by a distinguished surface by using another time the exponential charts; we can fullfill the triangle $\gamma_i, \gamma_j, \gamma_{\text{ref}}$ by a surface, because the finite energy loop space is simply connected. We poduce a distinguished surface $S_{i,j}(\gamma)$ with boundary $l_i(\gamma)$ and $l_j(\gamma)$. By using the theory of stochastic integrals, the integral of $\mu + \nu + \beta$ over $S_{i,j}(\gamma)$ exists almost surely (see [L9,L10]). Moroever, if we take the polygonal approximation $\gamma_.^n$ of $\gamma_.$, we have almost surely:

$$\int_{S_{i,j}(\gamma_.^n)} (\mu + \nu + \beta) \to \int_{S_{i,j}(\gamma)} (\mu + \nu + \beta). \tag{1.42}$$

If we imbed M into $\mathbb{R}^d$, we see that $\int_{S_{i,j}(\gamma)}(\mu + \nu + \beta)$ is the restriction of $\gamma \in B(\gamma_i, \delta)$ of a non-anticipative integral defined almost surely over the whole $L(M)$.

The condition $\gamma \in B(\gamma_i, \delta)$ is the first condition to define V_i (Condition $*$).

The second condition is $(\tau_1^Q)^{-1} \in G_i$ (Condition $**$).

The problem now is to lift this surface $S_{i,j}(\gamma)$ at the basis to $L_{\text{fin}}(Q)$ at the level of the total space. By the rule of calculus depending on a parameter, there exists a smooth version in t of $t \to \tau_1^Q(l_{i,t}(\gamma_.))$. There exists a section of $L_{\text{fin}}(Q)$ if $(\tau_1^Q)^{-1} \in G_i$: Let us call this section $q_{i,.}(\gamma_.)$. By pulling back the connection ∇^Q by using f and the commutative diagram (1.1), we get a lift of $l_{i,t}(\gamma_.)$ called $\tilde{l}_{i,t}(\gamma_.)$ starting from $q_{i,.}(\gamma_.)$ over the piece of the curve $l_{i,t}(\gamma_.)$ joining $\gamma_.$ to $\gamma_{i,.}$.

The next condition in order to define V_i (condition $***$) is that this occurs in a small neighborhood for the C^1 norm of a curve over $\gamma_{i,.}$: let us denote this curve $q_{i,.}$, by patching together the indices in the basis and in the fiber. We can therefore find a distinguished curve which joins the element of the path $\tilde{l}_{i,t}(\gamma_.)$ over $\gamma_{i,.}$ to the given element $q_{i,.}$ over $\gamma_{i,.}$: it is a vertical path; it is possible to do that because condition $***$ is checked. We go from $q_{i,.}$ to q_{ref} by any path over the path which goes from $\gamma_{i,.}$ to γ_{ref}. Conditions $*$, $**$, $***$ allow to produce a lift $\tilde{l}_{i,t}(q_{i,.}(\gamma_.))$ of $l_{i,t}(\gamma_.)$ starting from $q_{i,.}(\gamma_.)$ and arriving to q_{ref}. If $q_.$ lies in a small ball for the finite energy distance in the fiber (condition $****$) after the trivialization which arise from the local slice $\gamma_. \to q_{i,.}(\gamma_.)$, we can find a distinguished vertical path joining $q_.$ to $q_{i,.}(\gamma_.)$ and therefore a distinguished path joining $q_.$ to q_{ref} and which lifts $l_{i,t}(\gamma_.)$.

V_i is given by conditions $*$, $**$, $***$, $****$: condition $*$ allows to produce a distinguished curve joining $\gamma_.$ to γ_{ref}; condition $**$ allows to produce a section $q_{i,.}(\gamma_.)$ over $\gamma_.$; by using a connection, we produce a lift of the distinguished path joining $\gamma_.$ to γ_{ref}: condition $***$ allows to produce a vertical path joining the endpoint of this lift to $q_{i,.}$; condition $****$ allows to produce a distinguished vertical path joining $q_.$ to $q_{i,.}(\gamma_.)$.

The set of V_i clearly checks (i), because the parallel transport is almost surely defined and because almost surely the map $t \to \tau_1^Q(l_{i,t}(\gamma))$ is smooth in t.

Let us show that V_i checks (ii).

It is easy to find an approximation of the condition $**$ because $\gamma_. \to \tau_1^Q(\gamma_.)$ belongs to all the Sobolev spaces and because G is finite-dimensional.

Let us consider condition $*$. Let us proceed as in [JL2]. Let us introduce a function g from $[0, \infty[$ into $[1, \infty]$ which is equal to ∞ if $z > \delta$, which is equal to 1 if and only if $z \le \delta' < \delta'' < \delta$, and which behaves as $1/(\delta' - z)^{+n}$ (for a large negative integer), when $z \to \delta'_-$. Let F a smooth function from $[1, \infty[$ into $[0, 1]$, with compact support, and which is equal to 1 only in 1. Let F_i be the functional:

$$F_i = F\left(\int_0^1 g(\mathrm{d}(\gamma_s, \gamma_{i,s}))\,\mathrm{d}s\right), \tag{1.43}$$

F_i belongs to all Sobolev spaces of first order and produces an approximation of condition $*$.

We can perform the same approximation in the fiber by using:

$$F_i = F\left(\int_0^1 g(\mathrm{d}(g_s, g_{i,s}))\,\mathrm{d}s + \int_0^1 g(\mathrm{d}(g_s^{-1}\,\mathrm{d}/\mathrm{d}s g_s, g_{i,s}^{-1}\,\mathrm{d}/\,\mathrm{d}s g_{i,s}))\,\mathrm{d}s\right). \tag{1.44}$$

F this time is only equal to 1 in 2. We have used the section $q_{i,.}(\gamma_.)$ because condition $*$ is satisfied. This gives an approximation which belongs to all the Sobolev spaces in the fiber of condition $****$, by working as in [JL2] and in [L3], but with Eq. (1.15).

It remains to treat condition $***$ for the parallel transport. A sufficient condition is that the path $t \to \tau_1^Q(l_{i,t}(\gamma_.))$ remains in a small neighborhood of a given C^1 curve g_i in G. We choose for F_i:

$$F_i = F\left(\int_0^1 g(\mathrm{d}(\tau_1^Q(l_{i,s}(\gamma_.)), g_{i,s}))\,\mathrm{d}s\right.$$

$$\left. + \int_0^1 g(\mathrm{d}(\tau_1^Q(l_{i,s}(\gamma_.))^{-1}\,\mathrm{d}/\mathrm{d}s\tau_s^Q(l_{i,s}(\gamma_.)), g_{i,s}^{-1}\,\mathrm{d}/\mathrm{d}s\, g_{i,s}))\,\mathrm{d}s\right). \tag{1.45}$$

By using the fact that the modulus of continuity of $t \to \tau_1^Q(l_{i,t}(\gamma_.))$ and $t \to \tau_1^Q(l_{i,t}(\gamma_.))^{-1}$ $\mathrm{d}/\mathrm{d}t\tau_1^Q(l_{i,t}(\gamma_.))$ is known in probability, we deduce a regular approximation of condition $***$.

Therefore V_i satisfies (ii).

Let us now construct a random variable $\rho_{i,j}(q_.)$ with values in S^1 over $V_i \cap V_j$. Let us consider the lift $\tilde{l}_{i,t}(q_.)$ of $l_{i,t}(\gamma_.)$. It gives a curve which goes from $q_.$ to q_{ref}. If we can produce a distinguished surface $\tilde{S}_{i,j}(q_.)$ with boundary $\tilde{l}_{i,t}(q_.)$ and $\tilde{l}_{j,t}(q_.)$, this will allow us to produce a system of transition functions by integrating F_Q over this surface. This surface will be a lift of the basical surface $S_{i,j}(\gamma_.)$. Let us remark that in (i), there are four types of conditions. Especially $l_{i,t}(\gamma_.)$ can be equal to $l_{j,t}(\gamma_.)$ if $i \neq j$. But in all the cases, we can complete the loop constructed by $l_{i,t}(\gamma_.)$ and by $l_{j,t}(\gamma_.)$ runned in the opposite sense by a surface $S_{i,j}(\gamma_.)$ which can be treated as in (1.41) and (1.42).

We consider the path which joins $l_{i,t}(\gamma_.)$ to $l_{j,t}(\gamma_.)$ over the small triangle constituted by $\gamma_., \gamma_{i,.}, \gamma_{j,.}$. we call it $l_{i,j,t,u}(\gamma_.)$. We choose first a distinguished path joining $q_{i,.}(\gamma_.)$ to $q_{j,.}(\gamma_.)$. Then we lift the path $u \to l_{i,j,t,u}(\gamma_.)$ by starting from $\tilde{l}_{i,t}(q_.)$, and we arrive at $\bar{l}_{j,t}(q_.)$ over $l_{j,t}(\gamma_.)$. Therefore $\bar{l}_{j,t}(q_.)$ and $\tilde{l}_{j,t}(q_.)$ are in the same fiber. In a unique way:

$$\bar{l}_{j,t}(q_.) = \tilde{l}_{j,t}(q_.)g_{i,j,t}(q_.). \tag{1.46}$$

We join $q_{i,.}$ to $q_{j,.}$ by any deterministic curve, and we complete the triangle constituted by $q_{i,.}$, $q_{j,.}$, and q_{ref} by a deterministic curve. We complete the square constituted by $q_{i,.}(\gamma_.)$, $q_{j,.}(\gamma_.)$, $q_{i,.}$ and $q_{j,.}$ whose boundary is constituted by a loop in $L_{\mathrm{fin}}(G)$ by any surface in $L_{\mathrm{fin}}(G)$. We complete the loop $q_.$, $q_{i,.}$ and $q_{j,.}$ by any vertical loop in $L_{\mathrm{fin}}(G)$. The union of the two vertical surfaces which are obtained is called $\overline{S}_{i,j}(q_.)$. The surface which is obtained by lifting the path $u \to l_{i,j,t,u}(\gamma_.)$ is called $\tilde{S}'_{i,j}(q_.)$. We will not care of the deterministic surface which joins $q_{i,.}$, $q_{j,.}$ and q_{ref}. We put modulo this deterministic surface:

$$\tilde{S}_{i,j}(q_.) = \tilde{S}'_{i,j}(q_.) \cup \overline{S}_{i,j}(q_.). \tag{1.47}$$

We would like to define $\rho_{i,j}(q_.)$ by:

$$\rho_{i,j}(q_.) = \exp\left[-2\pi \mathrm{i} \int_{\tilde{S}_{i,j}(q_.)} F_q\right] \tag{1.48}$$

The boundary of the surface $\tilde{S}_{i,j}(q_.)$ consists of the two path $\tilde{l}_{i,t}(q_.)$ and $\tilde{l}_{j,t}(q_.)$ circled in the opposite sense. On the other hand, $\tilde{S}_{i,j}(q_.)$ projects over $S_{i,j}(\gamma_.)$: the only problem is for the deterministic surface completing the triangle $q_{i,.}$, $q_{j,.}$ and q_{ref} and the surface completing $\gamma_{i,.}$, $\gamma_{j,.}$ and γ_{ref}, but we can choose for the second one the projection of the first one. So the integral of the basical term can be treated by the method of [L9,L10].

Moreover

$$\tilde{l}_{i,j,t,u}(q_.)_s = \tau_s^Q(l_{i,j,t,u}(\gamma_.))g_s(l_{i,j,t,u}(q_.)) \tag{1.49}$$

is almost surely in $q_.$ smooth in t, u with values in $L_{\mathrm{fin}}(G)$ considered as a Hilbert–Lie group: in order to be precise, it is only piecewise smooth, but the subset where it is smooth do not depend of $q_.$. In particular if we look (1.49) only at the small triangle over $\gamma_.$, $\gamma_{i,.}$, $\gamma_{j,.}$, it is smooth in t and u. Therefore the vertical integral $\int_{\tilde{S}'_{i,j}(q_.)}(f^*)^*c$ is a traditional integral. It remains to consider the integral of $(f^*)^*c$ over the vertical surface $\overline{S}_{i,j}(q_.)$ which is a traditional integral.

Let us consider a polygonal approximation $q_.^n$ of $q_.$. This means that we take the polygonal approximation $\gamma_.^n$ of $\gamma_.$: this is possible if the length of the subdivision is large enough. We consider $q_s = \tau_s^Q g_s$, and we write $q_s^n = \tau_s^n(\gamma_.^n)g_s^n$ where g_s^n is a suitable polygonal approximation of g_s: the only problem to overcome is that $g_1^n = (\tau_1^Q(\gamma_.^n))^{-1}$ instead of $(\tau_1^Q(\gamma_.))^{-1}$, but it is not a real problem because these two elements of G are closed when n is large.

We have:

Lemma 1.8. *Almost surely over $V_i \cap V_j$*

$$\rho_{i,j}(q_.^n) \to \rho_{i,j}(q_.) \tag{1.50}$$

Proof. For the basical integral, this results from the rules of approximation of non-anticipative integrals.

For the vertical integral, the contribution of $\tilde{S}'_{i,j}(q^n_.)$ tends almost surely to the contribution of $\tilde{S}_{i,j}(q^n_.)$ because the parallel transport for the pullback connection depends smoothly from the curve $l_{i,j,t,u}(\gamma^n_.)$ (see Lemma 1.9 for similar considerations).

The main problem is the contribution of the integral of F_Q over $\overline{S}_{i,j}(q^n_.)$. But we can choose any vertical surface with the same boundary: its contribution changes by an integer and does not affect $\rho_{i,j}(q^n_.)$. In particular, we can choose a surface $\overline{S}_{i,j}(q^n_.)$ in $L_{\mathrm{fin}}(G)$ which is closed from $\overline{S}_{i,j}(q_.)$. □

Remark. We can give the general condition such that the set of transition functions can be defined. We produce a basical surface such that we can take the integral of $\mu + \nu + \beta$ by using the theory of Stratonovitch stochastic integral. We produce a lift of this basical surface such that the integral of $(f^*)^*c$ over the vertical component of this lift gives a traditional integral. There are of course a lot of these lifts, but, generally, the purpose of a connection is to lift a curve, therefore a surface. Moreover, the basical integral depends smoothly (in a generalized sense) on $\gamma_.$ as we will see, if the distinguished basical surface depends smoothly on $\gamma_.$ (in an appropriate sense which will be descibed in [L12]) and since the parallel depends smoothly on the curve, by using something as Lemma 1.9, the vertical part depends smoothly on $q_.$. This dependence of the transition functions works only in the Sobolev spaces and gives some rigidity to the notion of bundle (let us recall that all bundles over a finite-dimensional manifold are measurably trivial).

We can check now conditions (iv) and (v). F_Q is closed Z-valued over the finite energy-based loop space of Q. In particular, the transition function $\rho_{i,j}$ are defined for $q^n_.$ and check surely (iv) and (v). Lemma 1.8 implies (iv) and (v) for $\rho_{i,j}(q_.)$ almost surely.

It remains to check condition (iii). For this purpose, we introduce a small modification to the notion of Sobolev spaces at the basical level of $L(M)$ (see [L2,L3]). We consider $\gamma^n_.$ the polygonal approximation of $\gamma_.$, which exists modulo some boundary terms which are due to the fact that two continuous points over $\gamma_.$ γ_{s_i} and $\gamma_{s_{i+1}}$ can be far, given a suitable subdivision s_i of [0,1]. These boundary terms disappear when $n \to \infty$ (see [L2]). We take the derivative of cylindrical functionals for the vector fields $\tau^n_s H_s = X^n_s$ where τ^n_s is the parallel transport along the polygonal approximation, and we choose the horizontal lift $X^{\mathrm{h},n}_s$ of it. $\operatorname{div} X^{\mathrm{h},n}$ tends in law to $\operatorname{div} X^{\mathrm{h}}$ (see [L2]). So we decide in our definition of Sobolev spaces to take the derivatives of cylindrical functionals in the direction $X^{\mathrm{h},n}$ and to pass at the limit.

Let us consider $\rho_{i,j}(q^n_.)$. Since we can integrate over any surface $\tilde{S}_{i,j}(q^n_.)$ with boundary equal to the distinguished loop constituted by $\tilde{l}_{i,t}(q^n_.)$ and by $\tilde{l}_{j,t}(q^n_.)$ circled in the opposite sense, the derivative of $\rho_{i,j}(q^n_.)$ over $\tilde{X}^n$ is given by

$$\langle \mathrm{d}\rho_{i,j}(q^n_.), \tilde{X}^n \rangle = -2\pi \mathrm{i} \rho_{i,j}(q^n_.) \times \left(\int_{\tilde{l}_{j,t}(q^n_.)} F_Q(., \tilde{X}_t(\tilde{l}_{j,t}(q^n_.)) - \int_{\tilde{l}_{i,t}(q^n_.)} F_Q(., \tilde{X}_t(\tilde{l}_{j,t}(q^n_.))) \right) \tag{1.51}$$

$\tilde{X}_t(\tilde{l}_{i,t}(q_\cdot^n))$ is the derivative of $\tilde{l}_{i,t}(q_\cdot^n)$ along $\tilde{X}^n$: $\tilde{X}^n$ is a vector along $q_\cdot^n$ splitted in $X^{\mathrm{h},n} + X_{\mathrm{r}}^{\mathrm{v,n}}$.

There are two parts in (1.51): The part which is endowed with $\pi^*(\mu + \nu + \beta)$. It gives

$$\left(\int_{l_{j,t}(\gamma_\cdot^n)} - \int_{l_{i,t}(\gamma_\cdot^n)} \right) (\mu + \nu + \beta)(., X_t(l_{i,t}(\gamma_\cdot^n))). \tag{1.52}$$

By using the theory of non-anticipative Stratonovitch integrals and imbedding M into $\mathbb{R}^d$, the stochastic integral associated to (1.52) as well as its kernels tend in all L^p to

$$\left(\int_{l_{j,t}(\gamma_\cdot)} - \int_{l_{i,t}(\gamma_\cdot)} \right) (\mu + \nu + \beta)(., X_t(l_{i,t}(\gamma_\cdot))). \tag{1.53}$$

It remains the vertical part of (1.51).

Let us call $\tau_t^G(l)$ the parallel transport over a curve in G starting from a given element in $P_{\mathrm{fin}}(G)$. Let us consider a vector $K_s l_s$ along l.

The formula of [Bi1,Gr2] gives

$$\nabla^G_{K_\cdot l_\cdot} \tau_t^G(l) = \tau_t^G(l) \int_0^t (\tau_u^G(l))^{-1} R^G(\mathrm{d}l_u, K_u l_u) \tau_u^G(l)\, \mathrm{d}u, \tag{1.54}$$

where R^G is the curvature tensor for the connection ∇^G for the principal bundle $P_{\mathrm{fin}}(G) \to G$ whose structure group is reduced to the smooth based loop group.

We choose now the path $t \to (\tau_1^Q)^{-1}(l_{i,t}(\gamma_\cdot^n))$. We have:

$$\begin{aligned} &\nabla_{X^n} (\tau_1^Q)^{-1}(l_{i,t}(\gamma_\cdot^n)) \\ &\quad = K_t^n\ (\tau_1^Q)^{-1}(l_{i,t}(\gamma_\cdot^n)) \\ &\quad = -\int_0^1 (\tau_u^Q)^{-1}(l_{i,t}(\gamma_\cdot^n)) R^Q(\mathrm{d}_u l_{i,t}(\gamma_\cdot^n), X_{t,u}^n)(\tau_u^Q)(l_{i,t}(\gamma_\cdot^n))\ (\tau_1^Q)^{-1}(l_{i,t}(\gamma_\cdot^n)). \end{aligned} \tag{1.55}$$

K_t^n is smooth in t and tends in all the L^p to the limit expression K_t for the non-polygonal loop $\gamma_\cdot$. The main difficulty to treat in the vertical part of (1.51) is the derivative of the parallel transport over $l_{i,t}(\gamma_\cdot^n)$ in $L_{\mathrm{fin}}(Q)$ for a basical vector field X^n. The derivative along the fiber does not present so many difficulties, because we do not have to take the derivative of the parallel transport in the integration over the vertical paths joining $q_\cdot^n$ to $q_{i,.}(\gamma_\cdot^n)$ and joining $q_\cdot^n$ to $q_{j,.}(\gamma_\cdot^n)$.

The following lemma allows to solve the problem:

Lemma 1.9. *Uniformly in s and t if $*$, $**$ and $***$ are satisfied:*

$$\tau_t^G((\tau_1^Q)^{-1} l_{i,.}(\gamma_\cdot^n)) q_{i,.}(\gamma_\cdot^n)(s) \to \tau_t^G(\tau_1^Q)^{-1} l_{i,.}(\gamma_\cdot)) q_{i,.}(\gamma_{i,.})(s) \tag{1.56}$$

and

$$\mathrm{d}/\mathrm{d}s\, \tau_t^G((\tau_1^Q)^{-1} l_{i,.}(\gamma_.^n))q_{i,.}(\gamma_.^n)(s) \to \mathrm{d}/\mathrm{d}s\tau_t^G((\tau_1^Q)^{-1} l_{i,.}(\gamma_.))q_{i,.}(\gamma_.)(s) \tag{1.57}$$

in all the L^p.

Remark. s is the element of the typical path in $L_{\mathrm{fin}}(G)$ which is an element of the fiber.

Proof of Lemma 1.9. By cutting the time interval in a finite number of small subsets, we can suppose since $***$ is checked that the path $t \to (\tau_1^Q)^{-1}(l_{i,t}(\gamma_.^n))$ and $t \to (\tau_1^Q)^{-1}(l_{i,t}(\gamma_.))$ lies in a small neighborhood G_i of G where the path fibration $P_{\mathrm{fin}}(G) \to G$ is trivial. Let $K_{i,s}$ be the connection form of ∇^G over G_i.

Let us denote by l_t the path $t \to (\tau_1^Q)^{-1}(l_{i,t}(\gamma_.)$ and by l_t^n the path $t \to (\tau_1^Q)^{-1}(l_{i,t}(\gamma_.^n))$. Since $***$ is checked, we get

$$\tau_t^G(l_.)(g_.)(s) = \sum_m \int\limits_{0<t_1<\cdots<t_m<t} K_{i,s}(l_{t_1})(\mathrm{d}/\mathrm{d}t\, l_{t_1}) \cdots K_{i,s}(l_{t_m})(\mathrm{d}/\mathrm{d}t\, l_{t_m}) \times g_s\, \mathrm{d}t_1 \cdots \mathrm{d}t_m \tag{1.58}$$

and we get the analogous formula for $\tau_t^G(l_.^n)$. Eq. (1.58) is strongly based upon the fact that we consider the pullback connection of ∇^G over the loop space.

Let us denote by $I_m(l_.)$ an iterated integral of length m in (1.58) and by $I_m(l_.^n)$ the same iterated integral for $l_.^n$. We get on the other hand

$$\begin{aligned} I_m(l_.) - I_m(l_.^n) &= \sum_j \int\limits_{0<t_1<\cdots<t_m<t} K_{i,s}(l_{t_1}) K_{i,s}(l_{t_{j-1}})(\mathrm{d}/\mathrm{d}t\, l_{t_{j-1}}) \\ &\quad \times (K_{i,s}(l_{t_j})(\mathrm{d}/\mathrm{d}t\, l_{t_j}) - K_{i,s}(l_{t_j}^n)(\mathrm{d}/\mathrm{d}t\, l_{t_j}^n)) \cdots K_{i,s}(l_{t_m})(\mathrm{d}/\mathrm{d}t\, l_{t_m}) g_s^n\, \mathrm{d}t_1 \cdots \mathrm{d}t_m \\ &\quad + \int\limits_{0<t_1<\cdots<t_m<t} K_{i,s}(l_{t_1})(\mathrm{d}/\mathrm{d}t\, l_{t_1}) \cdots K_{i,s}(l_{t_m})(\mathrm{d}/\mathrm{d}t\, l_{t_m}) \\ &\quad \times (g_s - g_s^n)\, \mathrm{d}t_1 \cdots \mathrm{d}t_m, \end{aligned} \tag{1.59}$$

We get by [Bi1] or [Gr2]

$$\mathrm{d}/\mathrm{d}t\, l_t = \tau_1^Q(l_{i,t}(\gamma_.)) \int_0^1 (\tau_u^Q(l_{i,t}(\gamma_.)))^{-1} R^Q(\mathrm{d}_u l_{i,t}, \mathrm{d}/\mathrm{d}t(l_{i,t}(u)) \tau_u^Q(l_{i,t}(\gamma_.)) \tag{1.60}$$

and the analogous formula for $\mathrm{d}/\mathrm{d}t\, l_t^n$.

Let us consider an even integer p. An iterated integral of length m to the power p appears as at most C^m iterated integrals of length mp by the Stirling formula, because a product of iterated integrals is a sum of shuffle of iterated integrals of length equal to the sum of the length of each iterated integral; after this shuffle, a product of mp terms appears: p of these terms are tending to 0 and some of them are the discrete approximations of the integral and are polygonal approximations of (1.60), which are non-anticipative in the Stratonovitch sense. We distribute in a sum of at most $mp!$ iterated integrals of length smaller than $mp!$. For that we forget the contribution of $\tau_1^Q(l_{i,t}(\gamma_.^n))$ which is bounded because of ***. We have only to consider the contribution of $\int_0^1(\tau_u^Q)^{-1}(l_{i,t}(\gamma_.^n))R^Q(\mathrm{d}_u l_{i,t}(\gamma_.^n),\ \mathrm{d}/\mathrm{d}t\, l_{i,t}^n(u))\tau_u^Q)(l_{i,t}(\gamma_.^n))$. Let s_i be the time of the sudivision. We get

$$\int_{s_i}^{s_{i+1}} (\tau_u^Q)^{-1}(l_{i,t}(\gamma_.^n))R^Q(\mathrm{d}_u l_{i,t}(\gamma_.^n), \mathrm{d}/\mathrm{d}t\, l_{i,t}^n(u))(\tau_u^Q)(l_{i,t}(\gamma_.^n))$$
$$= F(\gamma_{s_i}, (\tau_{s_i}^Q)(\gamma_.^n), \gamma_{s_{i+1}}), \tag{1.61}$$

where $F = 0$ if $\gamma_{s_{i+1}} = \gamma_{s_i}$. Therefore F has bounded derivatives. We deduce that

$$\int_0^1 (\tau_u^Q)^{-1} l_{i,t}(\gamma_.^n)R^Q(\mathrm{d}_u l_{i,t}(\gamma_.^n), \mathrm{d}/\mathrm{d}t l_{i,t}^n(u))(\tau_u^Q)(l_{i,t}(\gamma_.^n)) = \int_0^1 \langle \alpha_n(s), \mathrm{d}\gamma_s\rangle, \tag{1.62}$$

where α_n is uniformly bounded when $n \to \infty$. Therefore our product of mp terms appears as the sum of at most $mp!$ iterated integrals of Stratonovitch type of bounded elements in all the Sobolev spaces. By using the Schwartz lemma [JL1], we see that the expectation of such stochastic iterated integrals is bounded by $C^{mp}/\sqrt{mp!}$. It remains to remark that a traditional deterministic integral of length mp is bounded by $C^m/mp!$ and since the series $\sum C^m/\sqrt{mp!}$, we deduce the first part of the lemma.

The second part is treated similarly, because in each product in (1.59), we have only to take the derivative of one term, which leads to m terms. The conclusion follows as before. □

1.4. Fourth step: Construction of stochastic gauge transforms

Let us denote by $\tilde{L}_{e,2}(G)$ the central extension of the based loop group with two derivatives in G in L^2.

Let us give the following definition:

Definition 1.10. A stochastic gauge transform of $L^\infty(\tilde{L}_{\mathrm{fin}}(Q))$ is a measurable application from $L(M)$ into $\tilde{L}_{e,2}(G)$.

The space of stochastic gauge transforms is a group for the natural composition over $\tilde{L}_{\mathrm{fin}}(G)$.

The main theorem of this part is the following:

Theorem 1.11. *The group of stochastic gauge transforms acts by isometries over $L^\infty(\tilde{\mu}_{\mathrm{fin}})$.*

Before proving this theorem, we will recall some statements of [CP,CM], in the smooth loop context.

An element of $\tilde{L}(Q)$ is the couple of a path starting from the constant reference path in Q and which arrives at $q_.$, and of an element of the circle.

Let us call

$$\tilde{l}_.(q_.) = (l_.(q_.), \beta) \tag{1.63}$$

this couple.

An element of $\tilde{L}(G)$ is the couple of a path starting from the unit path in G and arriving in $g_.$ and an element of the circle. Let us call

$$\tilde{l}_.(g_.) = (l_.(g_.), \alpha) \tag{1.64}$$

this couple. $\tilde{L}(G)$ acts on the right over $\tilde{L}(Q)$ by

$$\tilde{l}(q_.).\tilde{l}(g_.) = (l_.(q_.)_* l_.(g_.), \alpha\beta). \tag{1.65}$$

This means that after $q_.$, we continue the path $(q_.)$ by the path $q_. l_t(g_.)$ which is a vertical path.

Let a map from $L(M)$ into $\tilde{L}_{e,2}(G)$:

$$\gamma_. \to \tilde{l}_.(g_.(\gamma_.)) \tag{1.66}$$

let π be the projection of $L_{\text{fin}}(Q)$ in $L(M)$. Let F be a functional over $\tilde{L}_{\text{fin}}(Q)$. We associate the functional $F_{g_.(\gamma_.)}$:

$$\tilde{l}_.(q_.) \to F(\tilde{l}_.(q_.).\tilde{l}(g_.(\pi\gamma_.))). \tag{1.67}$$

The map $G_. : F \to F(\tilde{l}_.(q_.).\tilde{l}(g_.(\pi\gamma_.)))$ is a transformation from the space of measurable functionals over $\tilde{L}_{\text{fin}}(Q)$ into the space itself. The map $g_.(\gamma_.) \to G_.$ is a representation from the gauge group into the space of linear transformations of the space of functionals over $\tilde{L}_{\text{fin}}(Q)$.

Let us recall Lemma III.3 of [L10]. Let g_s^1 an element of $L_{e,2}(G)$ the loop space with two derivatives in $L^2$of G. The transformation

$$g_s \to g_s g_s^1 \tag{1.68}$$

transforms $\mathrm{d}P_g$ in a law which is equivalent. We can now give the proof of the theorem.

Proof of the Theorem 1.11. Over V_i, we have chosen a distinguished path $l_{i,.}(q_.)$. Let us suppose that $q_.g_.(\pi q_.)$ belongs to V_j. We have to consider the distinguished path $l_j(q_.g_.(\pi q_.))$. Moreover, we have the natural path joining $g_.(\pi q_.)$ to $e_.$ called $l'_.(g_.(\pi q_.))$. The paths $l_{j,.}(q_.g_.(\pi q_.))$ and $l_{i,.}(q_.)_* l'_.(g_.(\pi q_.))$ differ by a vertical surface. We can perform the integral of F_Q over this vertical surface. We find since this vertical can be chosen arbitrarly (only its boundary is given):

$$(l_{i,.}(q_.), \alpha).(l'_.(g_.(\pi q_.)), \beta) = l_{j,.}(q_.g_.(\pi\gamma_.)), \rho(q_.)\alpha\beta), \tag{1.69}$$

where $q_. \to \rho(q_.)$ is measurable.

The measure in the circle fiber is invariant by rotation, because we consider the Haar measure. Since the law of $q_{\cdot}g_{\cdot}(\pi\gamma_{\cdot})$ is equivalent to the law of $q_{\cdot}$ in the fiber, we deduce the result. □

2. The case of continuous loop groups

We consider still the commutative diagram (1.1), but we take for $P(G)$ the space of continuous paths in G starting from e and for $L(Q)$ the space of based continuous loops in Q over $L(M)$.

2.1. Construction of a measure over $L(Q)$

Let us consider this time the stochastic differential equation over G:

$$\mathrm{d}g_s = g_s\,\mathrm{d}B_s \tag{2.1}$$

starting from e, which plays here the role of (1.2). g_1 has a smooth density $q(g) > 0$, related to the heat semi-group over G. If we fix g_1, g_s is a semi-martingale going from e and arriving at g_1: its law is called $\mathrm{d}P_{g_1}$. Moreover if the law of g_s has the density $q_s(g)$, we get by the Markov property and the fact that g_s is a diffusion over a Lie group:

$$E_{g_1}[F(g_{s_1},\ldots,g_{s_r})] = \frac{\int q_{s_1}(g_1)q_{s_2-s_1}(g_1^{-1}g_2)\cdots q_{1-s}(g_{s_r}^{-1}g_1)\,\mathrm{d}\pi g_1\cdots\mathrm{d}\pi g_r}{q_1(g_1)}, \tag{2.2}$$

where π is the Haar measure over G. Eq. (2.2) plays the role of (1.8) of Definition 1.2.

We give the following definition which plays the role of Definition 1.2:

Definition 2.1. Over $L(Q)$, we define the measure by

$$\mathrm{d}\mu = \mathrm{d}P_{1.x}\otimes\mathrm{d}P_{(\tau_1^Q)^{-1}}. \tag{2.3}$$

We can give an expression of this measure in terms of cylindrical functionals $F(q_{s_1},\ldots, q_{s_r})$ with $s_1 < \cdots < s_r$:

$$\mu[F(q_{s_1},..q_{s_r})] = E_{1.x}[E_{(\tau_1^Q)^{-1}}[F(\tau_{s_1}^Q g_{s_1},..,\tau_{s_r}^Q g_{s_r})]]. \tag{2.4}$$

2.2. Second step: Construction of Sobolev spaces over $L(Q)$

Let us recall that the transition function of the principal bundle $P(G)\to G$ can be chosen in $L_{\mathrm{smooth}}(G)$, the space of based smooth loops in G. The connection form of ∇^G over G_i, a small neighborhood of G, where the bundle $P(G)\to G$ is trivialized is a one-form with values in $L_{\mathrm{smooth}}(\vartheta)$, the space of smooth based loops in the Lie algebra of G. It is denoted as in the first part $s\to K_{i,s}(\cdot)$.

We put if $(\tau_1^Q)^{-1}$ belongs in G_i, after trivialization of the fiber:

$$X_s^{\mathrm{h}} = \tau_s H_s - K_{i,s}(\langle \mathrm{d}(\tau_1^Q)^{-1}, X\rangle) g_s, \tag{2.5}$$

where $X_s = \tau_s H_s$ is a vector field over the basis in the sense of Bismut (see [Bi1]) (see (1.14) for the analogous formula, where we take $s \to g_s$ which is C^1 instead of being only continuous as here).

Since the connection form is intrisically defined, X^{h} is intrisically defined.

We get the analogous of Proposition 1.3.

Proposition 2.2. *Let F be a cylindrical functional over $L(Q)$. We get*

$$\mu[\langle \mathrm{d}F, X^{\mathrm{h}}\rangle] = \mu[F \operatorname{div} X^{\mathrm{h}}] \tag{2.6}$$

for a functional div X^{h} *which belongs to all the L^p if X^{h} is associated to $\tau_s H_s$ where h_s is deterministic.*

Proof. Let us recall the statement of the Albeverio–Hoegh-Krohn formula over a path group. Let g_s be the solution of (2.1) and g_s^1 a finite energy based loop in G. $g_s^1 g_s$ has a law which is absolutely continuous with respect of the law of g_s, with a Girsanov density which belong to all the L^p, and to all the Sobolev spaces. Namely:

$$\mathrm{d}(g_s^1 g_s) = \mathrm{d}g_s^1\, g_s + (g_s^1 g_s)\,\mathrm{d}B_s = g_s^1 g_s (g_s^{-1}(g_s^1)^{-1}\,\mathrm{d}g_s^1 g_s) + (g_s^1 g_s)\,\mathrm{d}B_s \tag{2.7}$$

$(g_s^1)^{-1}\,\mathrm{d}/\mathrm{d}s g_s^1$ is bounded in L^2; since $A \to (g_s)^{-1} A g_s$ is an isometry, we can apply the classical Girsanov theorem in order to conclude. The fact that the Girsanov density belongs to all the L^p is the important difference with the first part. Since $s \to K_{i,s}$ is a smooth application in s with values in the one-form, we deduce (2.6) as in Proposition 1.3. □

In order to define a vertical vector field, we consider the vector $g_s K_s$ as vector in the fiber instead of $K_s g_s$ as before, where $s \to K_s$ is a deterministic smooth based loop in the Lie algebra of G. We put in an intrinsic way:

$$X_{\mathrm{r}}^{\mathrm{v}} = q.K. \tag{2.8}$$

(see (1.19) and Proposition 1.4).

We get:

Proposition 2.3. *Let F be a cylindrical functional over $L(Q)$. We get*

$$\mu[\langle \mathrm{d}F, X_{\mathrm{r}}^{\mathrm{v}}\rangle] = \mu[F \operatorname{div} X_{\mathrm{r}}^{\mathrm{v}}], \tag{2.9}$$

where div $X_{\mathrm{r}}^{\mathrm{v}}$ *belongs to all the L^p.*

Proof. Instead of using the quasi-invariance formula of Albeverio–Hoegh-Krohn in the left, we use this formula in the right. Namely, with the same notation as in (2.7),

$$\mathrm{d}(g_s g_s^1) = g_s g_s^1 (g_s^1)^{-1}\,\mathrm{d}B_s g_s^1 + g_s\,\mathrm{d}g_s^1 \tag{2.10}$$

and $(g_s^1)^{-1}\,\mathrm{d}B_s g_s^1$ is still the differential of the Brownian motion in the Lie algebra. We use the quasi-invariance formula for $g_s^\lambda = \exp[\lambda K_s]$, we differentiate them and we desintegrate them. This gives (2.5). □

The tangent space of $T_{q.}(L(Q))$ is the Hilbertian orthogonal sum of the vertical vector fields $X_{\mathrm{r}}^{\mathrm{v}}$ and of the horizontal vector fields X^{h}. For a horizontal vector fields X^{h} associated to $\tau_s H_s$, we choose as Hilbert norm:

$$\|X^{\mathrm{h}}\|^2 = \int_0^1 \|\mathrm{d}/\mathrm{d}s H_s\|^2\,\mathrm{d}s. \tag{2.11}$$

For a horizontal vector field, $X_{\mathrm{r}}^{\mathrm{v}}$ associated to the finite energy loop K_s in the Lie algebra of G, we choose:

$$\|X_{\mathrm{r}}^{\mathrm{v}}\|^2 = \int_0^1 \|\mathrm{d}/\mathrm{d}s\; K_s\|^2\,\mathrm{d}s, \tag{2.12}$$

This shows us that the stochastic tangent space of $L(Q)$ is parallelisable: it is the sum of $L^2(T_x(M))$ and of $L^2(\vartheta)$ with conditions:

$$\int_0^1 \mathrm{d}H_s = 0, \qquad \int_0^1 \mathrm{d}K_s = 0. \tag{2.13}$$

The reader can see (1.23) and (1.24) for analogous formulas.

These integration by parts formulas allow us to close the operation of H-derivative (see [Gr1]) and to give the following definition.

Definition 2.4. $W_{1,p}(L(Q))$ is the space of functionals over $L(Q)$ endowed with the norm:

$$\|F\|_{W_{1,p}} = \|F\|_{L^p} + \|\,\|\mathrm{d}F\|\,\|_{L^p}, \tag{2.14}$$

where $\mathrm{d}F$ is the H-derivative of F and $\|\mathrm{d}F\|$ its Hilbert–Schmidt norm.

2.3. Third step: Functionals over the string bundle

We do the same hypothesis as in the first part.

Over $L(Q)$, we consider the form F_Q as in the first part, which is closed Z-valued. We would like to construct the space of the functionals over $\tilde{L}(Q)$, where $\tilde{L}(Q)$ is a formal circle bundle associated to F_Q. We produce a set V_i of measurable subsets of $L(Q)$ as in the first part. The only difference is that the vertical integral will be given by stochastic integrals, because c becomes a stochastic closed form. So we have to take care in the choice of the vertical surface.

For this reason, we have to take care in the choice of the distinguished neighborhood over $q_{i,.}(\gamma_.)$. We will overcome this problem by using in the fiber the same type of argument which were done in [L10] in order to construct over a loop space the space of L^p functionals of a formal circle bundle over it.

We choose a dense set of curves of smooth elements of $L(G)$, called g_i, and we choose the set of open balls $B(g_i, \delta)$ for δ small enough, which constitutes a cover of $L(G)$ for the uniform distance. If $g_. \in B(g_i, \delta)$, we deduce a distinguished path joining $g_.$ to $g_{i,.}$ by using a system of Lie group exponential charts (instead of Riemannian exponential charts as in [L9] or in [L10]). We choose any path joining $g_{i.}$ to $e_.$. V_i is determined as follows (we give the same index for the vertical condition and the horizontal condition): $q_. = q_{i,.}(\gamma_.)g_.$ where $g_. \in B(g_{i,.}, \delta)$. If $q_.$ belongs to V_i, we produce a distinguished path joining $q_.$ to $q_{i,.}(\gamma_.)g_{i,.}$ and to $g_{i,.}$ to $q_{i,.}(\gamma_.)$: then, we operate as in the first part. If $q_.$ is in V_i, the considered distinguished path which goes from $q_.$ to q_{ref} is called $\tilde{l}_{i,t}(q_.)$.

If $q_.$ belongs to $V_i \cap V_j$, we would like to produce a surface with boundary $\tilde{l}_{i,t}(q_.)$ and $\tilde{l}_{j,t}(q_.)$ where we should be able to integrate F_Q by using the theory of stochastic integrals. The only difference with the first part is when we have to take care to complete the small triangle $q_.$, $q_{i,.}(\gamma_.)g_{i,.}$ and $q_{j,.}(\gamma_.)g_{j,.}$ by a vertical surface, because the two-form c considered becomes a stochastic form in the fiber. Namely, we can join $q_{i,.}(\gamma_.)$ and $q_{j,.}(\gamma_.)g_{j,.}$ by a distinguished path $\tilde{l}_{i,j,t}(\gamma_.)$:

$$\tilde{l}_{i,j,t}(\gamma_.) = q_{i,.}(\gamma_.)g_{i,.}g_{i,j,t}(\gamma_.), \tag{2.15}$$

where $\tilde{l}_{i,j,0}(\gamma_.) = q_{i,.}(\gamma_.)g_{i,.}$ and $\tilde{l}_{i,j,t}(\gamma_.) = q_{j,.}(\gamma_.)g_{j,.}$.

We look at the part of the distinguished path $\tilde{l}_{i,t}(q_.)$ between $q_.$ and $q_{i,.}(\gamma_.)$. We apply $g_{i,j,u}(\gamma_.)(\gamma)$ to it. In u equal to 1, we arrive not far from the corresponding element for the corresponding path $\tilde{l}_{i,j}(q_.)$ between $q_.$ and $q_{j,.}g_{j,.}$. By using the Lie group exponential charts, we can complete the remaining hole by a distinguished surface. We construct by this procedure a vertical surface $S^2_{i,j}(q_.)$ which completes the triangle $q_.$, $q_{i,.}(\gamma_.)g_{i,.}$ and $q_{j,.}(\gamma_.)g_{j,.}$. As in the first part, we construct a surface which completes the triangle $q_{i,.}(\gamma_.)g_{i,.}$, $q_{j,.}(\gamma_.)g_{j,.}$ and q_{ref}. We call it $S^1_{i,j}(q_.)$. $S^1_{i,j}(q_.) \cup S^2_{i,j}(q_.)$ has boundary $\tilde{l}_{i,.}(q_.)$ and $\tilde{l}_{j,.}(q_.)$.

We have first to integrate F_Q over $S^1_{i,j}(q_.)$, which leads exactly to the same computations as in the first part. The only problem is to integrate F_Q over the vertical surface $S^2_{i,j}(q_.)$, that is c over $S^2_{i,j}(q_.)$.

It is formally a stochastic integral, but we have to explain a little bit what we mean by that. We can include G in $SO(n)$ such that we get a subbundle of a $SO(n)$ trivial bundle, and we can imbedd M into $\mathbb{R}^d$. Q is therefore a subset of $\mathbb{R}^d \times SO(n)$. Let us consider the triangle $(q_., q_{i,.}(\gamma_.)g_{i,.}, q_{j,.}(\gamma_.)g_{j,.})$: the surface $S_{i,j}(q_.)$ is given by the family $s \to l_{u,v}(q_s, s)$ which can be extended modulo the previous imbedding to all $q_.$. c can be extended over the based loop space of $\mathbb{R}^d$ because we can extend of g^{-1} from $T_g(G)$ over ϑ into an action from $T_x(\mathbb{R}^{d'})$ over $T_0(\mathbb{R}^{d'})$. We call $S^{1,\mathrm{ext}}_{i,j}(q_.)$ a surface in the loop space $\mathbb{R}^d \times SO(n)$ which extends to all the loop $q_.$ $S^1_{i,j}(q_.)$. We can integrate the extension of c by means of the theory of stochastic integrals over $S^{1,\mathrm{ext}}_{i,j}(q_.)$: this stochastic integral is not anticipative. It remains to introduce some cutoff functionals as in the first part in order to conclude.

As in the first part, we construct the transition functions over $V_i \cap V_j$ by

$$\rho_{i,j}(q_.) = \exp\left[-2\pi \mathrm{i} \int\limits_{S^1_{i,j}(q_.)\cup S^2_{i,j}(q_.)} F_Q\right], \tag{2.16}$$

which check (i)–(v).

Remark. The main difference with the first part is when we complete the small triangle $q_., q_{i.}, q_{j.}$ by a stochastic surface. This leads to a supplementary stochastic integral to treat, because the paths in the fiber are only semi-martingales.

We get the analogous definition for a measurable functional $\tilde{F}(\tilde{q}_.)$ of the formal circle bundle $\tilde{L}(\tilde{Q})$ as Definition 1.6. We can use the Definition 1.7 in order to define $L^p(\tilde{\mu})$, where $\tilde{\mu}$ is the formal measure over $\tilde{L}(\tilde{Q})$ which is the Haar measure in the fiber.

Definition 2.5. $L^{\infty-}(\tilde{\mu})$ is the intersection of all the $L^p(\tilde{\mu})$ $1 < p < \infty$.

2.4. Fourth step: Construction of stochastic gauge transforms

Let us consider the central extension $\tilde{L}_{e,1}(G)$ of finite energy loop in G. We get a map π from $\tilde{L}_{e,1}(G)$ over $L_{e,1}(G)$.

A stochastic gauge transform is a measurable map from $L(M)$ into $\tilde{L}_{e,1}(G)$: $\gamma_. \to \tilde{g}_.(\gamma)$.

Theorem 2.6. *Let G be a stochastic gauge transform $\gamma \to \tilde{g}_.(\gamma)$ such that $\pi\tilde{g}_.(\gamma)$ has a bounded energy when γ describes $L(M)$. The stochastic gauge transform induces a continuous application from $L^{\infty-}(\tilde{\mu})$ into $L^{\infty-}(\tilde{\mu})$.*

Proof. It preserves the integration in the fiber. Moreover the Girsanov density of the transformation $g_s \to g_s\pi\tilde{g}_s(\gamma_.)$ is bounded in $L^p(\mathrm{d}P_{(\tau_1^Q)^{-1}})$ when γ describes $L(M)$, because the loop $\pi\tilde{g}_.(\gamma_.)$ has a bounded energy when γ describes $L(M)$. □

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